

## SMEARING DISTRIBUTIONS AND THEIR USE IN FINANCIAL MARKETS

P. JIZBA\* and H. KLEINERT†

*Institute for Theoretical Physics, Freie Universität Berlin,  
Arnimallee 14 D-14195 Berlin, Germany*

\*E-mail: jizba@physik.fu-berlin.de

†E-mail: kleinert@physik.fu-berlin.de

It is shown that superpositions of path integrals with arbitrary Hamiltonians and different scaling parameters  $v$  (“variances”) obey the Chapman-Kolmogorov relation for Markovian processes if and only if the corresponding smearing distributions for  $v$  have a specific functional form. Ensuing “smearing” distributions substantially simplify the coupled system of Fokker-Planck equations for smeared and un-smeared conditional probabilities. Simple application in financial models with stochastic volatility is presented.

*Keywords:* Chapman-Kolmogorov equation; Path integrals; Heston’s model.

### 1. Introduction

One often encounters in practical applications probabilities formulated as a superposition (or “smearing”) of path integrals (PI) of the form

$$P(x_b, t_b | x_a, t_a) = \int_0^\infty dv \, \omega(v, t_{ba}) \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \mathcal{D}p \, e^{\int_{t_a}^{t_b} d\tau (ip\dot{x} - vH(p, x))}. \quad (1)$$

Here  $\omega(v, t_{ba})$  is some positive, continuous and normalizable function on  $\mathbb{R}^+ \times \mathbb{R}^+$  with  $t_{ba} = t_b - t_a$  being the time difference. Examples of (1) can be found in financial markets,<sup>1,2</sup> in polymer physics,<sup>1,3</sup> in superstatistics,<sup>4,5</sup> etc.

Whenever a smeared PI fulfills the Chapman-Kolmogorov equation (CKE) for continuous Markovian processes, the Feynman-Kac formula guarantees that such a superposition itself can be written as PI, i.e.

$$P(x_b, t_b | x_a, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \mathcal{D}p \, e^{\int_{t_a}^{t_b} d\tau (ip\dot{x} - \bar{H}(p, x))}. \quad (2)$$

The new Hamiltonian  $\bar{H}$  typically depends on first few momenta of  $\omega(v, t)$ .

Under normal circumstances, the smeared PI do not saturate CKE for Markovian processes, in fact, ad hoc choices of smearing distributions typically introduce memory into a dynamics. Our goal here will be to isolate the general class of continuous smearing distributions that conserve CKE. In Ref.<sup>7</sup> we have shown that ensuing distributions have important applications in evaluations of PI or in simplifications of the associated stochastic differential equations. In this note we shall briefly present the latter application. On the way we also mention a simple implication in economical models with stochastic volatility.

## 2. Most general class of smearing distributions

We look for  $\omega(v, t)$  fulfilling CKE for any intermediate time  $t_c$ :

$$P(x_b, t_b | x_a, t_a) = \int_{-\infty}^{\infty} dx P(x_b, t_b | x, t_c) P(x, t_c | x_a, t_a). \quad (3)$$

It can be shown<sup>7</sup> that Eq.(3) is fulfilled only when the integral equation

$$\int_0^z dz' \omega(z', t) a \omega\left(a(z - z'), \frac{t}{a}\right) = b \omega\left(bz, \frac{t}{b}\right), \quad (4)$$

( $a, b \in \mathbb{R}^+$  and  $1 + 1/a = 1/b$ ) holds. By defining the Laplace image function  $\tilde{\omega}$  as

$$\tilde{\omega}(\xi, t) = \int_0^{\infty} dz e^{-\xi z} \omega(z, t), \quad \Re \xi > 0, \quad (5)$$

Eq.(4) can be equivalently formulated as the functional equation for  $\tilde{\omega}$

$$\tilde{\omega}(\xi, t) \tilde{\omega}\left(\frac{\xi}{a}, \frac{t}{a}\right) = \tilde{\omega}\left(\frac{\xi}{b}, \frac{t}{b}\right). \quad (6)$$

Assumed normalizability and positivity of  $\omega(v, t)$  implies that smearing distributions are always Laplace transformable. After setting  $\alpha = 1/a$  we get

$$\tilde{\omega}(\xi, t) \tilde{\omega}(\alpha \xi, \alpha t) = \tilde{\omega}(\xi + \alpha \xi, t + \alpha t). \quad (7)$$

This equation can be solved by iterations. An explicit general solution of Eq.(7) was found in Ref.<sup>7</sup> and it reads

$$\tilde{\omega}(\xi, t) = \begin{cases} [G(\xi/t)]^t, & \text{when } t \neq 0, \\ \kappa^\xi, & \text{when } t = 0. \end{cases} \quad (8)$$

$G(x)$  is an arbitrary continuous function of  $x$ . Constant  $\kappa$  is determined through the initial-time value of  $\omega(v, t)$ . In particular,  $t \rightarrow 0$  solution (cf., Eq.(8)) gives

$$\lim_{\tau \rightarrow 0} \omega(v, \tau) = \theta(v + \log \kappa) \delta(v + \log \kappa) = \delta^+(v + \log \kappa). \quad (9)$$

Let us also notice that (7) implies  $\tilde{\omega}(\zeta, \tau) > 0$  for all  $\tau$  and  $\zeta$  which gives  $G(x) > 0$  for all  $x$ . This allows us to write

$$[G(\zeta/\tau)]^\tau = e^{-F(\zeta/\tau)\tau}. \quad (10)$$

Here  $F(x)$  is some continuous function of  $x$ . Final  $\omega(v, t)$  can be obtained via real Laplace's inverse transformation known as Post's inversion formula:<sup>6</sup>

$$\omega(v, t) = \lim_{k \rightarrow \infty} \frac{(-1)^k}{k!} \left( \frac{k}{v} \right)^{k+1} \frac{\partial^k \tilde{\omega}(x, t)}{\partial x^k} \Big|_{x=k/v}. \quad (11)$$

Real inverse transform (11) is essential because the solution of the functional equation (7) was found only for real variables in  $\tilde{\omega}(\xi, t)$ . In fact, complex functional equations are notoriously difficult to solve.

We finally point out that the result (8) is true also in the case when  $vH(p, x) = vH_1(p, x) + H_2(p, x)$ , such that  $[H_1, H_2] = 0$ .

### 3. Explicit form of $\bar{H}$

To find  $\bar{H}$  we use Post's formula (11) which directly gives

$$\begin{aligned} & P(x_b, t_b; x_a, t_a) \\ &= - \lim_{k \rightarrow \infty} \int_0^\infty dx \frac{(-x)^{k-1}}{(k-1)!} \frac{\partial^k \tilde{\omega}(x, t)}{\partial x^k} \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \mathcal{D}p e^{\int_{t_a}^{t_b} d\tau (ip\dot{x} - kH/x)} \\ &= \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \mathcal{D}p \left[ \int_0^\infty dy \omega(y, t) e^{-y \int_{t_a}^{t_b} d\tau H} \right] e^{i \int_{t_a}^{t_b} d\tau p \dot{x}} \\ &= \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \mathcal{D}p e^{\int_{t_a}^{t_b} d\tau (ip\dot{x} - F(H))}. \end{aligned} \quad (12)$$

In passing from second to third line we have used the asymptotic expansion of the modified Bessel function  $K_k(\sqrt{k}x)$  for large  $k$ . On the last line we have utilized the definition of the Laplace image. Result (12) thus allows to identify  $\bar{H}$  with  $F(H)$ . Let us finally mention that the normalization

$$1 = \int_0^\infty dv \omega(v, t) = \tilde{\omega}(0, t) \quad (13)$$

implies that  $F(0) = 0$ .

#### 4. Kramers-Moyal expansion for $\omega(v, t)$

Let us now mention a simple application in stochastic processes. To this end we notice that both  $P(x_b, t_b|x_a, t_a)$  and

$$P_v(x_b, t_b|x_a, t_a) \equiv \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \mathcal{D}p \, e^{\int_{t_a}^{t_b} d\tau (ip\dot{x} - vH)}, \quad (14)$$

fulfil CKE, so they can be alternatively evaluated by solving Kramers-Moyal's (KM) equations:<sup>1,8</sup>

$$\frac{\partial}{\partial t_b} P(x_b, t_b|x_a, t_a) = \mathbb{L}_{KM} P(x_b, t_b|x_a, t_a), \quad (15)$$

$$\frac{\partial}{\partial t_b} P_v(x_b, t_b|x_a, t_a) = \mathbb{L}_{KM}^v P_v(x_b, t_b|x_a, t_a), \quad (16)$$

where the KM operator  $\mathbb{L}_{KM}$  is

$$\mathbb{L}_{KM} = \sum_{n=1}^{\infty} \left( -\frac{\partial}{\partial x_b} \right)^n D^{(n)}(x_b, t_b), \quad (17)$$

and similarly for  $\mathbb{L}_{KM}^v$ . The coefficients  $D^{(n)}$  and  $D_v^{(n)}$  are defined through the corresponding short-time transitional probabilities, so e.g.,  $D_v^{(n)}(x, t)$  is

$$D_v^{(n)}(x, t) = \frac{1}{n!} \lim_{\tau \rightarrow 0} \frac{1}{\tau} \int_{-\infty}^{\infty} dy (y - x)^n P_v(y, t + \tau|x, t). \quad (18)$$

Equations (16) can be cast into an equivalent (and more convenient) system of equations. For this we rewrite (4) as

$$\omega(z, t) = \int_0^{\infty} dz' \omega(z', t') P^{\omega}(z, t|z', t'), \quad (19)$$

with the conditional probability

$$P^{\omega}(z, t|z', t') = \frac{t}{t - t'} \theta(tz - t'z') \omega\left(\frac{tz - t'z'}{t - t'}, t - t'\right),$$

$$\int_0^{\infty} dz P^{\omega}(z, t|z', t') = 1, \quad \lim_{\tau \rightarrow 0} P^{\omega}(z, t + \tau|z', t) = \delta^+(z - z'). \quad (20)$$

Eqs. (19)–(20) ensure that the transition probability  $P^{\omega}(z, t|z', t')$  obeys CKE for a Markovian process. Since the process is Markovian, we can define KM coefficients  $K^{(n)}$  in the usual way as

$$K^{(n)}(v, t) = \lim_{\tau \rightarrow 0} \frac{1}{n! \tau} \int_{-\infty}^{\infty} dx (x - v)^n P^{\omega}(x, t + \tau|v, t). \quad (21)$$

From ensuing CKE for a short-time transition probability one may directly write down the KM equation for  $\omega(v, t_{ba})$

$$\frac{\partial}{\partial t_{ba}} \omega(v, t_{ba}) = \mathbb{L}_{KM}^{(\omega)} \omega(v, t_{ba}), \quad \mathbb{L}_{KM}^{(\omega)} = \sum_{n=1}^{\infty} \left( -\frac{\partial}{\partial v} \right)^n K^{(n)}(v, t_{ba}). \quad (22)$$

In cases when both (16) and (22) are naturally or artificially truncated at  $n = 2$  one gets two coupled Fokker-Planck equations

$$\begin{aligned} \frac{\partial}{\partial t} \omega(v, t) &= \mathbb{L}_{FP}^{(\omega)} \omega(v, t), \quad \frac{\partial}{\partial t_b} P_v(x_b, t_b | x_a, t_a) = \mathbb{L}_{FP}^v P_v(x_b, t_b | x_a, t_a), \quad (23) \\ \mathbb{L}_{FP}^{(\omega)} &= \mathbb{L}_{KM}^{(\omega)}(n = 1, 2), \quad \mathbb{L}_{FP}^v = \mathbb{L}_{KM}^v(n = 1, 2). \end{aligned}$$

On the level of sample paths the system (23) is represented by two coupled Itô's stochastic differential equations

$$\begin{aligned} dx_b &= D_v^{(1)}(x_b, t_b) dt_b + \sqrt{2D_v^{(2)}(x_b, t_b)} dW_1, \\ dv &= K^{(1)}(v, t_{ba}) dt_{ba} + \sqrt{2K^{(2)}(v, t_{ba})} dW_2. \end{aligned} \quad (24)$$

Here  $W_1(t_b)$  and  $W_2(t_{ba})$  are respective Wiener processes.

## 5. Economical models with stochastic volatility

Note that in (24) the dynamics of the variance  $v$  is explicitly separated from the dynamics of  $x_b$ . This is a desirable starting point, for instance, in option pricing models.<sup>1</sup> As a simple illustration we discuss the stochastic volatility model presented in Ref.<sup>2</sup> To this end take  $G$  to be

$$G(x) = \left( \frac{b}{x+b} \right)^c, \quad b \in \mathbb{R}^+; \quad c \in \mathbb{R}_0^+. \quad (25)$$

This gives

$$\tilde{\omega}(\zeta, t) = \left( \frac{bt}{\zeta + bt} \right)^{ct} \Rightarrow \omega(v, t) = \frac{(bt)^{ct} v^{ct-1}}{\Gamma(ct)} e^{-btv}. \quad (26)$$

$F(0) = 0$  as it should. Distribution (26) corresponds to the Gamma distribution<sup>2,9</sup>  $f_{bt,ct}(v)$ . The Hamiltonian  $\bar{H}$  associated with (26) reads

$$\bar{H}(p, x) = \bar{v} b \log \left( \frac{H(p, x)}{b} + 1 \right), \quad (27)$$

where  $\bar{v} = c/b$  is the mean of  $\omega(v, t)$ . As  $H$  we use the Hamiltonian from Refs.<sup>1,2</sup> which has the form  $\mathbf{p}^2/2 + i\mathbf{p}(r/v - 1/2)$ ,  $r$  is a constant. This choice ensures that  $P_v(x_b, t_b | x_a, t_a)$  represents a riskfree martingale distribution.<sup>1</sup> Full discussion of this model without truncation is presented in Ref.<sup>2</sup>

We consider here the truncated-level description that is epitomized by Itô's stochastic equations (24). The corresponding drift and diffusion coefficients  $D_v^{(1)}$  and  $D_v^{(2)}$  are then (cf. Eq.(18))

$$D_v^{(1)}(x, t_b) = \left(r - \frac{v}{2}\right), \quad D_v^{(2)}(x, t_b) = \frac{v}{2}. \quad (28)$$

For the coefficients  $K^{(n)}$  an explicit computation gives

$$K^{(1)}(v, t_{ba}) = \frac{1}{t_{ba}} (\bar{v} - v), \quad K^{(2)}(v, t_{ba}) = \frac{1}{t_{ba}^2} \frac{c}{2b^2}. \quad (29)$$

Consequently, the Itô's system takes the form

$$dx_b = \left(r - \frac{v}{2}\right) dt_b + \sqrt{v} dW_1, \quad dv = \frac{1}{t_{ba}} (\bar{v} - v) dt_{ba} + \frac{1}{t_{ba}} \sqrt{\frac{\bar{v}}{b}} dW_2. \quad (30)$$

Let us now view  $x_b$  as a logarithm of a stock price  $S$ , and  $v$  and  $r$  as the corresponding variance and drift. If, in addition, we replace for large  $t_{ab}$  the quantity  $\sqrt{\bar{v}}$  with  $\sqrt{v}$ , the systems (30) reduces to

$$dS = rS dt_b + \sqrt{v}S dW_1, \quad dv = \gamma (\bar{v} - v) dt_{ba} + \varepsilon \sqrt{v} dW_2. \quad (31)$$

The system of equations (30) corresponds to Heston's stochastic volatility model,<sup>1,10</sup> with mean-reversion speed  $\gamma = 1/t_{ba}$  and volatility of volatility  $\varepsilon = 1/(t_{ba}\sqrt{b})$ .

### Acknowledgments

This work has been partially supported by the Ministry of Education of the Czech Republic under grant MSM 6840770039, and by the Deutsche Forschungsgemeinschaft under grant Kl256/47.

### References

1. H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets*, 4th edn. (World Scientific, 2004).
2. P. Jizba, H. Kleinert, and P. Haener, [arXiv:0708.3012] .
3. A. Kholodenko, *Ann. Phys.* **202**, p. 186 (1990).
4. F. Sattin, *Physica A* **338**, p. 437 (2004).
5. C. Beck and E. Cohen, *Physica A* **322**, p. 267 (2003).
6. E. Post, *Trans. Amer. Math. Soc.* **32**, p. 723 (1930).
7. P. Jizba and H. Kleinert, FU-Berlin preprint, to appear shortly
8. N. V. Kampen, *Stochastic Processes in Physics and Chemistry*, 2nd edn. (North Holland, 1993).
9. W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. II*, 2nd edn. (John Wiley, 1966).
10. S. Heston, *Trans. Amer. Math. Soc.* **6**, p. 327 (1993).